Topics on social and economic networks

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Chapter 2

Diffusion of Behavior in Network Games Orchestrated by Social Learning

2.1 Introduction

The interest in understanding the driving forces of collective behavior has triggered an extensive literature that spans economics, sociology, marketing and epidemiology. In their seminal work, Schelling (1971) and Granovetter (1978) develop models of collective behavior for situations where individuals have two alternatives and the costs and/or benefits of each depend on how many individuals choose which alternative. Examples are numerous, including riot behavior, innovation and rumor diffusion, strikes, consumption network externalities, spread of fashions, etc. The key element of Schelling (1971) and Granovetter (1978) is the concept of a threshold, i.e., the number or proportion of others who must take action before a given individual does so, or equivalently, the point at which net benefit begins to exceed net cost for that particular individual. A characteristic of these models is that a particular action alternative will only be adopted on a large scale if it achieves some critical mass of support. A major drawback of these models, however, is that they don't provide an explanation where this critical mass comes from as it is usually treated as an exogenous factor. In the context of diffusion of innovation Jackson and Yariv (2005, 2007) offer the metaphor of a free trial period of new technology. In his models of rioting behavior Kuran (1989, 1991) suggests that the early mobilization problem will only be resolved through a catalytic event, a "spark", that reveals the hidden unpopularity of the present regime. On the outbreak of the revolutions in Eastern Europe and the French Revolution he notes, however, that this spark is often difficult to discern, explaining why these occurrences seem to "appear out of nowhere".

Another drawback of Schelling (1971) and Granovetter (1978) is that the structure of communication among individuals is not explicitly modeled; implicitly the seminal models assume that every individual is informed about the decision of everybody else. Jackson and Yariv (2005, 2007) showed how to extend Granovetter's model with respect to social structure. Individuals are connected according to an undirected graph (the social network) and decide whether to switch to a new behavior, e.g. a new technology, or stay with the default alternative. The return to each action depends on the number of neighbors in the network, the proportion of neighbors adopting the new alternative and some individual-specific cost-benefit parameters. Individuals are supposed to be myopic and play best responses to the neighbors' action in the previous period. Similar to the classical works their model features the existence of a tipping point, or in other words an unstable equilibrium, which is largely determined by the network. Although Jackson and Yariv successfully give an answer to the effect of social network structures, the question remains about the *endogenous* triggers and driving forces of collective action. As noted in Valente (2005) in the context of diffusion of innovation, "Verbal accounts on how people make decisions and adopt behavior usually reveal ...whims that are not independent of networks, but not easily captured in social influence models." The state-of-the-art literature is not capable of explaining the hidden driving force of collective action as an endogenous property of dynamics.

In his overview article about innovation diffusion, Young (2009) compares three broad classes of models: contagion, social influence, and social learning. He summarizes the approach of social learning as "People adopt once they see enough empirical evidence to convince them that the innovation is worth adopting, where the evidence is generated by the outcomes among prior adopters. Individuals may adopt at different times due to differences in their prior beliefs, amount of information gathered, and idiosyncratic costs." Hence people want to see how it works for others over a period of time before trying themselves. As he points out it is difficult to summarize the sizable literature on social learning due to the great diversity of informational assumptions. He nevertheless identifies the dynamic characteristics of a fairly general class of learning models which has a surprisingly simple structure. It boils down to Bayesian updating of beliefs about the quality of innovation relative to the status quo. Individuals have initial beliefs about the payoff of the innovation, based on partial information, and update their beliefs through a random meeting process with adopters. The updated belief is the weighted average of initial belief and the average payoff of the observations. Individuals adopt the innovation when their updated beliefs exceed their costs. Effects of the social network architecture are not discussed as he implicitly assumes a complete communication network.

In this chapter, we combine the two approaches of collective action on networks and social learning in order to model collective action with endogenous driving force. Our model is therefore described by two factors: one corresponding to *learning about* the benefits of adoption (the invisible part) and one corresponding to *best replying to* the benefit of adopting the alternative action (the visible action part). The goal is to analyze the dynamic behavior of such a coupled system and to understand the influence of the network.

We follow and extend Jackson and Yariv (2005, 2007) in working with a stylized model of network games. Each individual has to make a binary decision: a choice between actions A and B. Here action A is the status quo and action B is an alternative. An individual switches to action B if it appears worthwhile of doing so. The decision depends on the costs and benefits of switching, where the payoff is influenced by the number of neighbors of the individual's that have already switched. The adoption rate of the alternative action is the only variable in Jackson and Yariv (2005, 2007). We add another variable, the *inclination*, into the model. The inclination is the tendency of an individual to adopt the new behavior. It can be seen as the inner driving force of decision making, and therefore captures the "sentiment" part described in the literature. We formalize individual inclination updating process by the social learning model in Young (2009). This means individual inclination is also influenced by the inclination and behavior of the people who share social connections with the individual.

Each individual is characterized by the number of her neighbors, namely her degree, and her costs where the latter are randomly and independently assigned before the process starts. Exact local network structure cannot be efficiently used for large networks because of the complexity of the problem, thus we use averaged information instead of exact information. This "mean-field approach" is done by assuming that individuals only have the information about the degree distribution and cost distribution of the population, but don't know with whom they are connected. In other words, there are no fixed neighbor sets. This idea is quiet different from traditional network theory using exact graph structures, e,g. Galeotti and Goyal (2009) and Grabisch and Rusinowska (2010, 2011, 2013), among others, where the interdependencies between individuals are fixed. To motivate it, one may imagine a dynamic decision making problem in discrete time where two time epochs are considered in each period, i.e. the beginning and the end of that period. At the beginning of a period, individuals need to make decision which takes place at the end of that period. It is often the situation that at the beginning of the period, an individual does not know with whom she will interact during the period, but only knows about the number of individuals she will connect. Nevertheless, a decision must be made at the beginning of the period, which is possibly because of the need of a budget. At the end of the period, all information is revealed and used for the next period. This way of aggregating information in modelling makes the model tractable, especially for large social networks where exact structure is hardly obtained. On the other hand, roughness is always accompanied with this approach. We note that the matching phase of individuals in each period is not explicitly considered. For convenience, one can think of an appropriate random matching mechanism.

With our model we are able to explain a variety of behavior of the adoption rates by incorporating endogenous inclination dynamics. Of particular interest is the existence of non-monotonic behavior of the aggregated adoption rate, which is not possible under the original model of Jackson and Yariv (2005, 2007). Our abstract model and findings can be taken as a metaphor for many applications. For example, in sociology, it provides a step towards understanding the volatility of rioting behavior and the impact of the social network architecture. In marketing, our results contribute to understanding why a new product becomes a success or failure. In financial markets, the results could advance the understanding of market sentiments.[1]

The structure of this chapter is as follows. Section 2.2 describes the strategic choice of adoption as a best response dynamics when the inclinations are given as a parameter and vary exogenously. Section 2.3 introduces social learning as an endogenous process of inclination formation and shows how to couple learning to the best response dynamics of Section 2.2. It also discusses equilibrium existence and structure for a selection of networks. Other dynamic properties are illustrated by numerical simulation. Section 2.4 concludes the chapter. Proofs of propositions are collected in Section 2.5.

2.2 Diffusion dynamics on networks

2.2.1 The basic model

We consider a society of individuals, each of them chooses an action between two alternatives A and B. Assume A is the default behavior (the status quo). Choosing B can be interpreted as adopting a new technology, or learning another language, etc. The underlying social network structure is given and characterized by the degree distribution $P(d)$ for $d \in \mathscr{D}$ where $\mathscr{D}$ is the set of all degrees and $\sum_{d \in \mathscr{D}} P(d) = 1$.[2] We assume that $0 \notin \mathscr{D}$, i.e., there is no isolated individual in the network.

In this chapter, we characterize a network by means of its degree distribution, although in commonly accepted social network analysis this is only a characteristic of a network. Individual i's degree is denoted by d_i. We assume that each individual only knows how many neighbors she has, but does not know who those neighbors are.[3] Such a situation can arise

[1] Here, we think of "sentiment" of an investor as the attitude of investors as to anticipated price developments in a market, and of "market sentiment" as the general prevailing attitude.

[2] For a given network, $P(d)$ is calculated as the fraction of nodes that have degree d. It also means that the probability of a random node in the network having degree d is $P(d)$.

[3] The total number of individuals in the network is not known to anyone.

when the decision to adopt is based on previous observations. In that case, each individual has a certain estimation of the number of people she will interact with in the future. The set of neighbors of each individual is treated as if it was randomly selected from the rest of the population according to the degree distribution $P(d)$. Put

$$\widetilde{P}(d) = \frac{P(d)d}{\overline{d}}, \quad \overline{d} = \sum_{k \in \mathscr{D}} P(k)k. \tag{2.1}$$

Here, $\widetilde{P}(d)$ represents the probability that a random chosen neighbor of a random individual in the network has degree d where $\overline{d}$ denotes the average degree of the network.[4] It is easy to see that $\sum_{d \in \mathscr{D}} \widetilde{P}(d) = 1$, which means $\widetilde{P}(d)$ is a probability distribution. From the definition we also see that $\widetilde{P}(0) = 0$, which is reasonable since if an individual is a neighbor of someone else, she cannot have degree 0.

Individual i's utility from adopting action $a \in \{A, B\}$ is given by $u(a, d_i, x_i)$, where d_i is her degree and x_i is her estimation of the fraction of individuals in the population who has already switched to B. Let $v(d_i, x_i)$ be defined as

$$v(d_i, x_i) = u(B, d_i, x_i) - u(A, d_i, x_i), \tag{2.2}$$

representing the benefit of switching from A to B.[5] Each individual i has an idiosyncratic cost $c_i > 0$ of switching which we will specify shortly. Individual i switches to action B if her benefit is higher than her cost, i.e.,

$$v(d_i, x_i) > c_i. \tag{2.3}$$

Let $v(d, x) = g(d)x$ where $g(d)$ is a function capturing how the number of neighbors affects the individual benefit of adopting B. For example, if $g'(d) > 0$ then individuals with higher degrees have higher benefit and so are more likely to adopt than those with less neighbors. With this functional form the adoption rule (2.3) reads

$$g(d_i)x_i > c_i. \tag{2.4}$$

We assume the costs c_i are randomly and independently assigned to each individual i according to a probability distribution. Following Jackson and Yariv (2005), the cost distribu-

[4] For a derivation of (2.1) see Newman (2010, Chapter 13.3).

[5] It sounds plausible to let the utility of individual i depend on the number of adopters of action B in her neighbor set. But since the neighbor set is treated as a randomly selected set of individuals, this number of adopters in the neighbor set equals the number of neighbors times the overall fraction of adopters of B in the whole population estimated by the individual. Hence the setting here is appropriate.

tion is indirectly given by the distribution of the reciprocal cost. This implies that for each individual we consider the random variable $1/c_i$ instead of c_i. More precisely, denote $C \geq 0$ the random variable of individual cost, and let $F : \mathbb{R}_+ \to [0,1]$ be the cumulative distribution function of $1/C$. Rearranging (2.4) provides

$$\frac{1}{c_i} > \frac{1}{g(d_i)x_i}. \tag{2.5}$$

Individual i's probability of choosing action B is therefore

$$\Pr\left[v(d_i, x_i) > c_i\right] = \Pr\left[\frac{1}{c_i} > \frac{1}{g(d_i)x_i}\right] = 1 - \Pr\left[\frac{1}{c_i} \leq \frac{1}{g(d_i)x_i}\right] = 1 - F\left(\frac{1}{g(d_i)x_i}\right). \tag{2.6}$$

For mathematical simplicity, we assume that F is continuous and twice differentiable.

Remark 2.1. (i) Each individual individual i knows only her own degree d_i and her cost c_i. In other words, individuals only have partial information about the degrees and costs of the population, given by distributions P and F. Thus, the model corresponds to a Bayesian game in the Harsanyi sense (Harsanyi (1967, 1968a,b)) where the types of the game are given by vector (d_i, c_i).
(ii) Inequality (2.4) can be rewritten as

$$x_i \geq \frac{c_i}{g(d_i)}. \tag{2.7}$$

Here the fraction $c_i/g(d_i)$ on the right-hand side can be thought as the threshold of individual i for adopting action B. In the classic model of Granovetter (1978), the "threshold" is "the number or proportion of others who must make one decision before a given actor does so" (see also Schelling (1971)).

2.2.2 Dynamics and equilibria

Here we define the discrete time process of adoption rate. In the remaining of this chapter, a variable in period t is expressed with a superscript (t). At $t = 0$, a given fraction $x^{(0)}$ of the population is exogenously switched to adopt action B. At each time $t > 0$, each individual, *including those individuals whose initial adoption is B*, uses the actual aggregated adoption rate in period $t-1$ as her estimated (or believed) adoption rate of her neighbor set in the current period t, and makes her decision of adoption. Denote the aggregated adoption rate in period t by $x^{(t)}$.

We assume that the individuals are myopic, best respond to the state of the last period. Let $x_d^{(t)}$ denote the fraction of individuals who have degree d and have adopted action B in period t. Then, for each $t > 0$,

$$x_d^{(t)} = 1 - F\Big(\frac{1}{g(d)x^{(t-1)}}\Big), \tag{2.8}$$

where $x^{(t)}$ represents the link-weighted average of $x_d^{(t)}$, i.e.,

$$x^{(t)} = \sum_{d \in \mathscr{D}} \widetilde{P}(d) x_d^{(t)}. \tag{2.9}$$

Equations (2.8) and (2.9) can be interpreted as follows. All individuals with the same degree are indistinguishable, therefore they have the same probability of adopting action B, as described in Equation (2.8). It is assumed that individuals do not use the information based on their own experience in the past, e.g. $x_d^{(t-1)}$, because the population is large and most degrees are usually small which makes this information not useful in interacting with new neighbors. Therefore, the calculation of probability to adopt B is based on the aggregated adoption rate of the previous period for all individuals. For aggregated adoption rate $x^{(t)}$, since individuals cannot specify their neighbors, the estimation of $x^{(t)}$ is based on expectation. A random neighbor with degree d has probability $\widetilde{P}(d)$ to be chosen, and the chosen neighbor has in turn a probability of $x_d^{(t)}$ to adopt action B at time t, hence the expected adoption rate $x^{(t)}$ is the weighted average of $x_d^{(t)}$ as shown in Equation (2.9). We assume $x^{(t)}$ is announced to every individual at the end of each period $t \geq 0$.

By combining (2.8) and (2.9) and using the property $\sum_{d \in \mathscr{D}} \widetilde{P}(d) = 1$, one has

$$x^{(t)} = 1 - \sum_{d \in \mathscr{D}} \widetilde{P}(d) F\Big(\frac{1}{g(d)x^{(t-1)}}\Big). \tag{2.10}$$

By defining

$$h(x) := 1 - \sum_{d \in \mathscr{D}} \widetilde{P}(d) F\Big(\frac{1}{g(d)x}\Big), \tag{2.11}$$

one can derive the equilibria of adoption rate as solutions of equation

$$x = h(x). \tag{2.12}$$

2.2.3 Diffusion process

Jackson and Yariv (2005) analyze the model for $g(d) = \alpha \cdot d^{\beta}$ where α and β are constants .[6] The parameter α can be seen as a key factor of individual thresholds. It amplifies or decreases the effect of individual degrees on the overall adoption rate. As described in Equation (2.7), a larger α implies a lower individual threshold, which implies a higher individual *inclination* to adopt action B. Here we say that individuals have higher inclination if their probability of adopting action B is higher given that their degrees and the adoption rate of their neighbors (hence the overall adoption rate) are the same. From Equation (2.6), one sees that a higher α results in a higher probability of adoption. Therefore we call α the inclination parameter and later on treat it as a variable of the model.

In our generalized model we will drop the assumption that α is a static parameter and allow it to change over time. The idea is to introduce a process of social learning. People observe empirical evidence from prior adopters in their neighborhood which influences their inclinations towards adopting the innovation. This translates into the model as a dynamic variable $\alpha^{(t)}$ (α at time t) which is determined endogenously by a learning process. Our purpose is to study its evolution and in particular how its behavior affects the dynamics of adoption.

To get some feeling for the influence of inclination $\alpha^{(t)}$ on the adoption rate $x^{(t)}$, we first discuss some simple cases where $\alpha^{(t)}$ is either constant or exogenously determined. The explicit learning dynamics will be introduced in Section 2.3.

Diffusion process with constant α

Example 2.1. Assume that $g(d) = \alpha$ for every $d \in \mathscr{D}$, so that the benefit from choosing action B does not depend on individual degrees. Here, Equation (2.8) simplifies to

$$x_d^{(t)} = 1 - F\left(\frac{1}{\alpha x^{(t-1)}}\right) = x^{(t)}.$$

Consider a uniform inverse cost distribution F on $[0, b]$, $b > 0$. By definition, the uniform distribution on $[0, b]$ has cumulative distribution function

$$F(y) = \begin{cases} 0 & \text{for } y < 0, \\ y/b & \text{for } y \in [0, b], \\ 1 & \text{for } y > b. \end{cases} \tag{2.13}$$

[6] As an extension, Jackson and Yariv (2007) treated $v(d, x)$ as a general function without explicitly considering its shape.

Function $h(x)$ describing the dynamics of process $\{x^{(t)} \mid t \geq 0\}$ given in (2.11) then takes the following simple form

$$h(x) = 1 - F\Big(\frac{1}{\alpha x}\Big) = \begin{cases} 0 & \text{for } x < \frac{1}{b\alpha}, \\ 1 - \frac{1}{b\alpha x} & \text{for } x \geq \frac{1}{b\alpha}. \end{cases} \tag{2.14}$$

The dynamics of $\{x^{(t)} \mid t \geq 0\}$ depends on the number of intersections between $h(x)$ and the 45° identity line as illustrated in Figure 2.1. We see that $h(x)$ intersects the 45° identity line twice for α large enough. If so, the lower interception point is an unstable equilibrium, usually referred to as a "tipping point".[7] If the dynamics start above the tipping point and below the upper intersection point, the adoption rate $x^{(t)}$ increases monotonically to the upper intersection point, which is a stable equilibrium. If the dynamics start above this stable equilibrium, $x^{(t)}$ decreases monotonically to it. Below the tipping point, the dynamics of $x^{(t)}$ decrease monotonically to zero. Closed forms of the equilibria easily follow from the fixed point equation $h(x) = 1 - \frac{1}{b\alpha x} = x$. If there is only one intersection, i.e., the identity line is tangent to function $h(x)$, the intersection is a semi-stable equilibrium. For α small enough, no intersection exists for $x > 0$. In this case the adoption rate converges to zero regardless of where it starts.

Example 2.2 (continued from Example 2.1)**.** Now instead of uniform distribution we assume that F is a normal distribution with mean μ and variance σ^2 given by $F(y) = \Phi(y \mid \mu, \sigma)$. For $g(d) = \alpha$, function $h(x)$ is expressed by

$$h(x) = 1 - \Phi\Big(\frac{1}{\alpha x} \,\Big|\, \mu, \sigma\Big), \quad 0 \leq x \leq 1. \tag{2.15}$$

This is the setting of the classical threshold model of Granovetter (1978). Figure 2.2 illustrates the dynamics for different values of α. Again, changing α can result in different types of equilibrium, implying a variety of behavior of the adoption rate $x^{(t)}$.

The simplicity of Example 2.1 and 2.2 originates from the assumption of $g(d)$ being constant in d such that (2.11) simplifies considerably and the degree distribution, i.e. network structure, $P(d)$ has no influence on the dynamical behavior. If $g(d)$ is a function of degrees, however, the dynamics becomes more complicated in the sense that $h(x)$ is typically only piecewise differentiable. The reason is that the adoption rule in (2.5) generates different results for different degrees, such that individuals with different degrees start adopting earlier or later. The network structure $P(d)$ averages over different adoption rates, therefore leaving it's footprint on the dynamics of $x^{(t)}$.

[7] See e.g. Jackson and Yariv (2007).

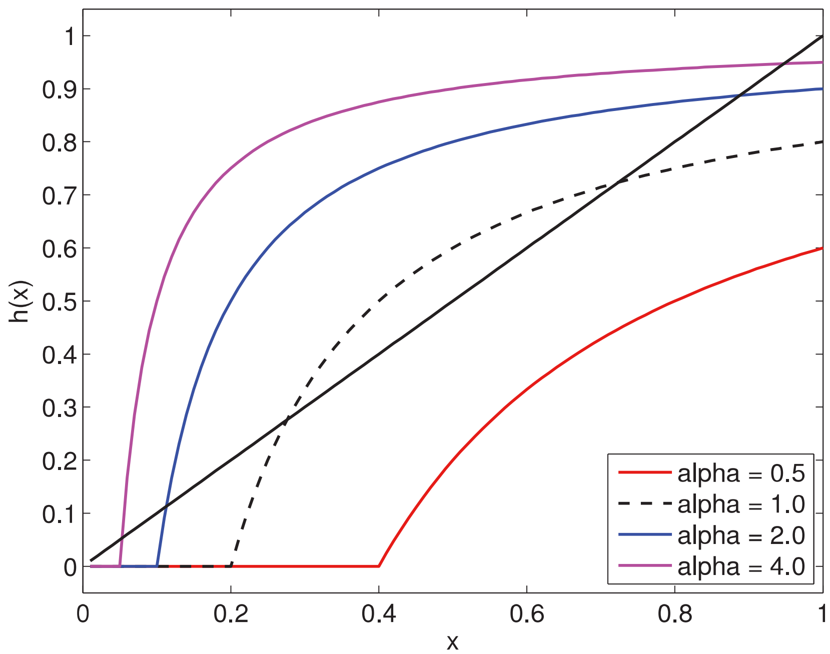


Figure 2.1: Dynamics of $x^{(t)}$ with $\alpha = 0.5, 1, 2, 4$ and $F \sim \text{Uniform}[0, 5]$.

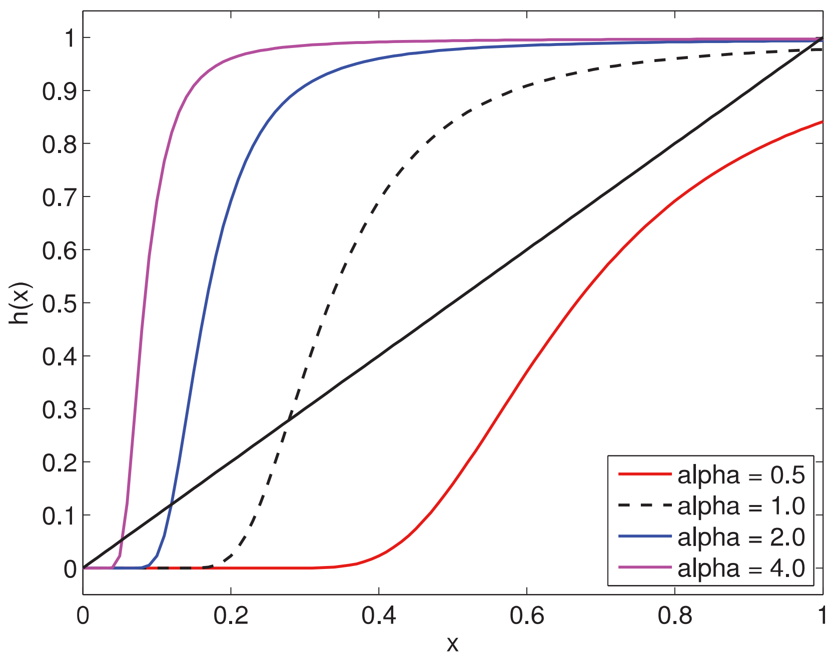


Figure 2.2: Dynamics of $x^{(t)}$ under $F \sim N(\mu = 3, \sigma = 1)$ with $\alpha = 0.5, 1, 2, 4$

Proposition 2.1. *Suppose that F is uniform on* $[0, b]$ *and* $g(d) > 0$. *Then* $h(x)$ *is non-decreasing, continuous, and piecewise concave.*

Proof. Let $S_d \in [0,1]$ denote the level of overall adoption that starts triggering adoption of degree d individuals. In other words, S_d is the smallest value of aggregated adoption rate $x^{(t)}$ such that the probability of adopting B for individuals with degree d in period $t+1$ is positive. Precisely,

$$S_d = \inf\left\{x : 1 - F\Big(\frac{1}{g(d)x}\Big) > 0\right\} = \max\left\{x : 1 - F\Big(\frac{1}{g(d)x}\Big) = 0\right\}.$$

Hence when x passes the points S_d from below, a new class of individuals with degree d starts joining, which raises the rate of increase of $h(x)$.

Let $(k_1, \ldots, k_{|\mathcal{D}|})$ be a permutation of $\mathcal{D}$ such that $S_{k_i} \le S_{k_j}$ if $i < j$. From (2.11) it follows that

$$\begin{aligned} h(x) &= 1 - \sum_{d\in\mathcal{D}} \widetilde{P}(d) F\Big(\frac{1}{g(d)x}\Big) = \sum_{d\in\mathcal{D}} \widetilde{P}(d)\Big\{1 - F\Big(\frac{1}{g(d)x}\Big)\Big\} \\ &= \sum_{d\in\mathcal{D}} \widetilde{P}(d) \times h_d(x) = \sum_{i=1}^{|\mathcal{D}|} \widetilde{P}(k_i) \times h_{k_i}(x). \end{aligned}$$

In particular, when F follows the uniform distribution on $[0, b]$, the response dynamics of degree d individuals is given by

$$h_d(x) := 1 - F\Big(\frac{1}{g(d)x}\Big) = \begin{cases} 0 & \text{for } x < S_d, \\ 1 - \frac{1}{bg(d)x} & \text{for } x \ge S_d, \end{cases} \tag{2.16}$$

which is strictly increasing, continuous and concave for $x \ge S_d$. Since $h_d(x) = 0$ for $x < S_d$, one has

$$h(x) = \begin{cases} \sum_{i=1}^{|\mathcal{D}|} \widetilde{P}(k_i) h_{k_i}(x) & \text{when } x \ge S_{k_{|\mathcal{D}|}}, \\ \sum_{i=1}^{j} \widetilde{P}(k_i) h_{k_i}(x) & \text{when } S_{k_j} \le x < S_{k_{j+1}} \text{ for } 1 \le j < |\mathcal{D}|, \\ 0 & \text{when } x < S_{k_1}. \end{cases} \tag{2.17}$$

We conclude that $h(x)$ consists a weighted sum of non-decreasing, continuous and concave functions and thus inherits these properties. □

Remark 2.2. From Figure 1 in Jackson and Yariv (2005) it is tempting to jump to the conclusion that the effect of replacing $g = \alpha$ by $g(d) = \alpha d^{\beta}$ with $\beta > 0$ is merely an upwards shift in adoption levels. However, the effect on the dynamics can be very complicated. For instance,

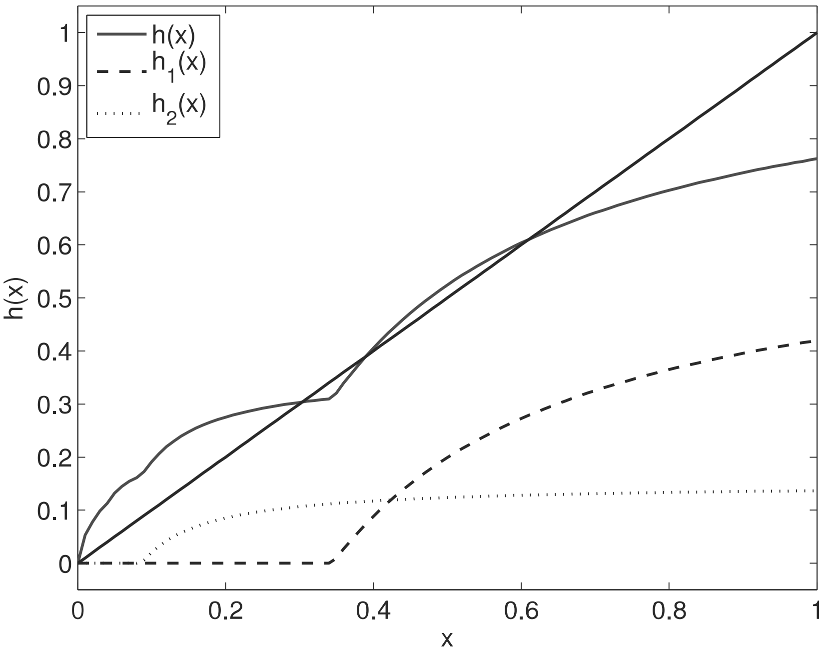


Figure 2.3: Dynamics of $x^{(t)}$ with $\beta = 2$, $\alpha = 1$, $P(d)$ proportional to $d^{-3.1}$, and $F \sim$ Uniform$[0, 2.9]$. In this figure, the solid blue line represents $h(x)$ which is the weighted sum of $h_i(x)$ for $i \geq 1$. To provide a better vision, only $h_1(x)$ and $h_2(x)$ are depicted in addition to $h(x)$.

multiple stable equilibria as well as tipping points may exist as shown in following example.

Example 2.3. Assume $g(d) = \alpha d^{\beta}$ with $\alpha > 0, \beta > 0$ and suppose F is uniform on $[0, b]$. Let $\mathscr{D} = \{1, \ldots, D\}$, Equation (2.17) then holds with $k_i = D - i + 1$, $i \in \mathscr{D}$. Figure 2.3 gives an illustration for $\beta = 2$. The dashed lines $h_1(x)$ and $h_2(x)$ show the adoption behavior for degree 1 and 2 individuals. The graph of $h_2(x)$ starts increasing at around 0.08, meaning that 8% overall adoption rate starts triggering the adoption of individuals with degree 2. Individuals with the lowest degree 1 are the last to join adoption with positive values of $h_1(x)$ starting at around 0.34. The total number of adopters is the sum of the numbers of adopters with different degrees. As shown in Figure 2.3, function $h(x)$ can have multiple stable and unstable equilibria.

Diffusion process with time varying $\alpha^{(t)}$

We now discuss the speed of convergence, especially for the situation that the initial adoption rate $x^{(0)}$ is between a tipping point and the next higher stable equilibrium. For constant α and $\beta = 0$ Jackson and Yariv (2007) shows that the adoption rate over time exhibits an S-shape (see their Proposition 7). In other words, the speed of convergence accelerates first

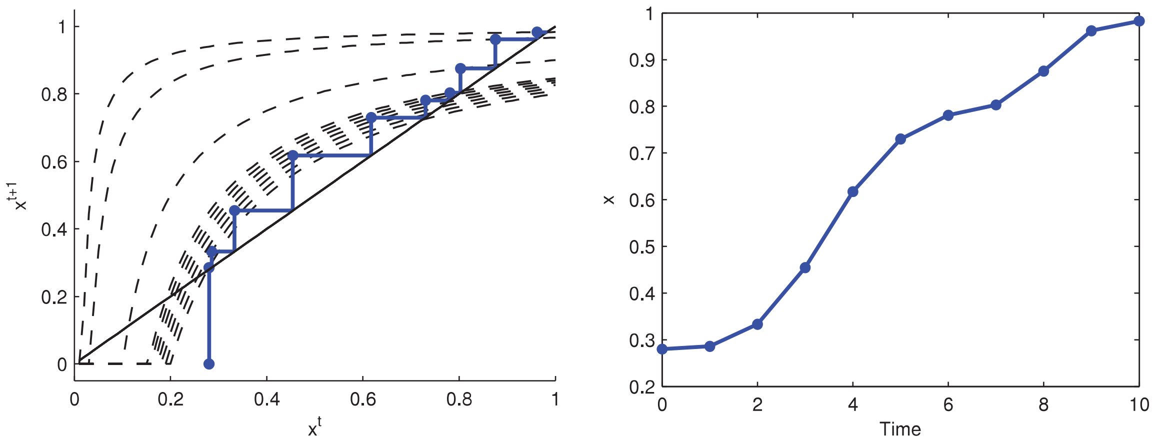


Figure 2.4: Dynamics of $\{x^{(t)} : t = 0, \ldots, 10\}$ with $x^{(0)} = 0.28$, where $\alpha^{(t)} = 1$, 1, 1.05, 1.1, 1.15, 1.2, 1.25, 1.3, 2.0, 6.0, 12.0 for $t = 0, \ldots, 10$, and $b = 5$.

and then decelerates after the adoption rate passing a critical value. This property is not necessarily preserved under the model with time varying $\alpha^{(t)}$. The next examples illustrates some effects on the speed of convergence.

Example 2.4. Suppose $\beta = 0$ and let $\alpha^{(t)}$ take values 1, 1, 1.05, 1.1, 1.15, 1.2, 1.25, 1.3, 2.0, 6.0, 12.0 for $t = 0, \ldots, 10$ respectively. Figure 2.4 illustrates the evolution of $x^{(t)}$. Up to $t = 7$ the increasing speed of $\alpha^{(t)}$ is constant and $x^{(t)}$ indeed changes from accelerating to decelerating. For $t > 7$, however, the rapid change of $\alpha^{(t)}$ results in a vigorous shift of $h(x)$ to the upper-left which leads again to an acceleration of $x^{(t)}$. In conclusion, the speed of convergence of $x^{(t)}$ depends on the dynamics of $\alpha^{(t)}$.

Example 2.5. Now consider the effect of α changing over time, say, $\alpha^{(t)}$ is monotonically increasing in t defined by $\alpha^{(t)} = \alpha^{(0)} + \sum_{s=1}^{t}(1/2)^s$ with $\alpha^{(0)} = 1/2$. Interestingly, the effect on the adoption rate $x^{(t)}$ can be non-monotonic as illustrated by Figure 2.5. The dynamics start at low levels such that $x^{(t)}$ moves towards zero. However, since $\alpha^{(t)}$ "heats up" in the background, the graph of $h(x \mid \alpha^{(t)})$ shifts to the top left quickly enough to "catch" the current level of $x^{(t)}$. In other words, individual thresholds of adopting B decrease quickly enough such that the tipping point falls below $x^{(t)}$ before it drops to zero.

In addition to the affect on convergence speed, Example 2.5 shows that the evolution of aggregated adoption rate $x^{(t)}$ can be even non-monotonic. It seems that this non-monotonicity comes from the artificially given $\alpha^{(t)}$, but we will show it is rather a systemic phenomenon in the study of endogenous $\alpha^{(t)}$ in the next section.

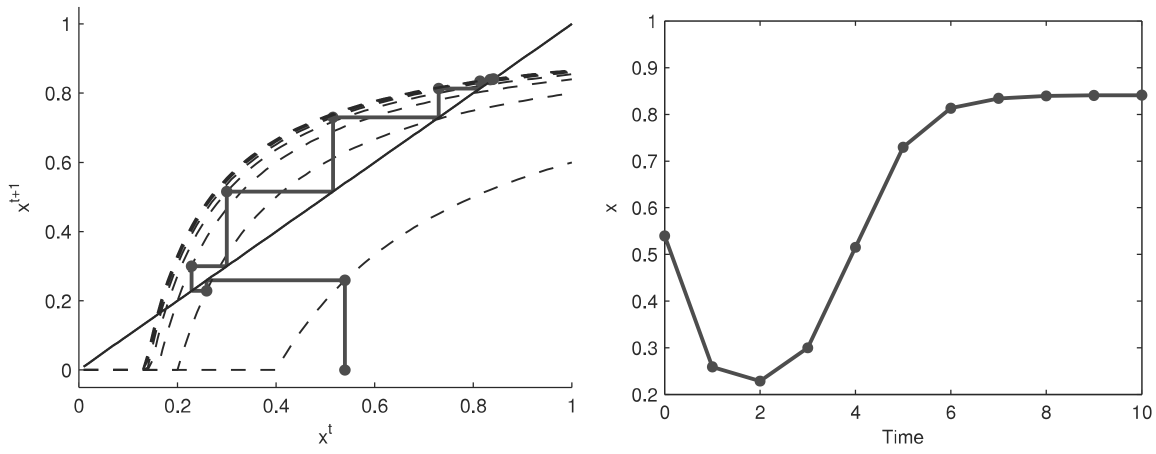


Figure 2.5: Dynamics of $\{x^{(t)} : t = 0, \dots, 10\}$ with $x^{(0)} = 0.54$, where $\alpha^{(t)} = \alpha^{(0)} + \sum_{s=1}^{t} (1/2)^s$ for $t = 1, \dots, 10$, $\alpha^{(0)} = 1/2$, and $F \sim \text{Uniform}[0,5]$. The dynamics follows $x^{(t+1)} = 1 - F\big(\frac{1}{\alpha^{(t)} x^{(t)}}\big)$.

2.3 Model the evolution of inclination by social learning

In this section we consider a class of social learning models in which individuals rationally evaluate the observed evidence of prior adopters. The idea is that when an innovation – like e.g. a new technology or a new management strategy – becomes available, people want to see how it works for others before trying themselves. As Young (2009) points out the literature on social learning is extensive. By means of some simplifying assumptions, however, he introduces a fairly general class of social learning models that still allows for heterogeneous characteristics. We will reintroduce his general framework by adding the feature that learning is governed by the social network.[8] The next step will be to couple the learning mechanism to the diffusion model discussed in Section 2.2.

2.3.1 The model

We consider individual inclination parameter $\alpha_d^{(t)}$ also depends on individual degree d, and incorporate a learning process into $\alpha_d^{(t)}$ which makes inclination evolve endogenously. Thus, $\alpha_d^{(t)}$ becomes a variable of the system rather than a parameter. Let the aggregated inclination at time $t \geq 0$ be given by

$$\alpha^{(t)} = \sum_{d \in \mathcal{D}} \widetilde{P}(d) \alpha_d^{(t)} \,. \tag{2.18}$$

[8]Young (2009) models flow of information by a Poisson arrival process including a parameter measuring the extent to which an individual “gets around”, which could be interpreted as a degree in a social network. He notes, however, that this parameter will not be sufficient to describe impact of the network topology on the dynamics. (See his footnote 16.)

Consider now the function $g(d)$ is time dependent through $\alpha_d^{(t)}$, i.e., let $g(d) = \alpha_d^{(t)} d^\beta$ at time t. It captures how the number of neighbors affects the individual benefits from adopting B.[9] For each individual of degree d let $\alpha_d^{(0)} \in (0, \infty)$ denote the initial inclination towards switching behavior. As time proceeds information from adopters keeps coming in and individuals update their inclinations.

For $t > 0$, each individual will have met a random draw of neighbors at $t-1$. If an individual has d neighbors and the overall fraction of adopters is $x^{(t-1)}$, the expected number of independent observations of adopters is $dx^{(t-1)}$, and her updated inclination $\alpha_d^{(t)}$ is defined as the weighted average of her own initial inclination $\alpha_d^{(0)}$ and the old aggregated inclination $\alpha^{(t-1)}$, i.e.,

$$\alpha_d^{(t)} = \frac{\tau_d \alpha_d^{(0)} + dx^{(t-1)} \alpha^{(t-1)}}{\tau_d + dx^{(t-1)}} \quad \text{for} \quad d \in \mathscr{D}\,, \tag{2.19}$$

where $\tau_d \in (0, \infty)$ reflects *flexibility* in learning (see Groot (1970), Young (2009)). Low values of τ_d indicate that relatively little evidence is necessary to change individual i's inclination.[10] The term $dx^{(t-1)}$ is the expected number of neighbors who have adopted B. It might seem ridiculous to use an expected value for a period in the past, where exact values should be available. Recall that individuals are recognized by their degrees, i.e. people with the same degree are treated as the same, thus no information about the actual number of neighbors for each individual can be used in the model. So in this mean-field approach, we use expected number instead of exact number. $dx^{(t-1)}\alpha^{(t-1)}$ expresses the amount of influence that an individual receives from her neighbors, where $\alpha^{(t-1)}$ is the averaged influence per individual. Note that there is no updating if the term $dx^{(t-1)}$ is zero, i.e. there is no learning in absence of evidence.

It is now an easy task to combine the learning dynamics (2.19) with the dynamics of adoption as discussed in the previous section. Substituting $g(d) = d^\beta \alpha_d^{(t-1)}$ into Equation (2.10) gives the update formula for aggregated adoption rate

$$x^{(t)} = 1 - \sum_{d \in \mathscr{D}} \widetilde{P}(d) F\Big(\frac{1}{d^\beta \alpha_d^{(t-1)} x^{(t-1)}}\Big). \tag{2.20}$$

[9]Young (2009) assumes that $g(d) = Ad^\beta$, where A is a normally distributed random variable with mean $\mu > 0$ and variance σ^2, independent and identically distributed among individuals and time periods. Here, μd^β is the mean individual payoff gain per period of switching to B. Ex ante, however, individuals are not informed about the true value of μ and start with different inclinations at $t = 0$. In order to demonstrate dynamic effects of initial expectations we leave the initial levels as a degree of freedom.

[10]This definition corresponds to an *infinite memory property*, which means that every individual remembers her initial inclination and updates it forever. There are other possible definitions, e.g., replacing $\tau_d \alpha_d^{(0)}$ by $\tau_d \alpha_d^{(t-1)}$.

Now (2.19) and (2.20) formulate a $D+1$ dimensional system of $\big(\{\alpha_d^{(t)}\}_{d\in\mathscr{D}}, x^{(t)}\big)$ where the initial values $\{\alpha_d^{(0)}\}_{d\in\mathscr{D}}$ remain in the system for every $t>0$. Because of this feature, this system cannot be analyzed as a normal dynamical system.

For notational simplicity, we express $\{y_d\}_{d\in\mathscr{D}}$ by $\{y_d\}$ if there is no ambiguity. Let $H: \mathbb{R}_+^D \times [0,1] \to \mathbb{R}_+^D \times [0,1]$ denote the right hand side of (2.19) and (2.20), i.e.,

$$H\left(\begin{matrix}\{\alpha_d\}\\ x\end{matrix}\,\middle|\,\{\alpha_d^{(0)}\}\right) := \begin{bmatrix}\left\{\frac{\tau_d\alpha_d^{(0)}+dx(\sum_{d\in\mathscr{D}}\widetilde{P}(d)\alpha_d)}{\tau_d+dx}\right\}\\ 1-\sum_{d\in\mathscr{D}}\widetilde{P}(d)F\big(\frac{1}{d^\beta\alpha_d x}\big)\end{bmatrix}. \tag{2.21}$$

Then (2.19) and (2.20) can be simplified to

$$\begin{bmatrix}\{\alpha_d^{(t)}\}\\ x^{(t)}\end{bmatrix} = H\left(\begin{matrix}\{\alpha_d^{(t-1)}\}\\ x^{(t-1)}\end{matrix}\,\middle|\,\{\alpha_d^{(0)}\}\right). \tag{2.22}$$

What we are interested in is the dynamics of aggregated inclination and adoption rate $(\alpha^{(t)}, x^{(t)})$, which is a projection from $D+1$ dimensional space onto a two dimensional space.

2.3.2 Types of equilibrium

A state is said to be an equilibrium of the $D+1$ dimensional system (2.22) if the system starts at that state and remains there forever. Precisely, $(\{\alpha_d\}, x)$ is an equilibrium of (2.22) if

$$(\{\alpha_d^{(t)}\}, x^{(t)}) = (\{\alpha_d\}, x) \quad \text{for all} \quad t \ge 0.$$

Since no individual inclination is changed when the system is in equilibrium, this equilibrium is an *individual* equilibrium. Later on we will discuss *aggregated* equilibrium where only aggregated inclination and adoption rate are considered.

In absence of adoption, i.e. $x=0$, there is no learning from adopters and hence no individual inclination is ever updated as indicated by (2.19). With respect to adoption rate note that the right-hand side of (2.20) is zero if $x=0$. Hence the best response to a zero adoption level is again zero. We shall refer to equilibria with zero adoption rate as *trivial* equilibria.

Can an individual equilibrium be non-trivial? The answer is yes. It easily follows from (2.19) that for $x>0$ this is the case if and only if $\alpha_d^0 = \alpha^0$ for all $d\in\mathscr{D}$. The intuition is straightforward. If inclinations are identical there is no learning. We shall refer to this equilibrium simply as *identical* equilibrium. In fact, non-identical and non-trivial individual equilibria do not exist. Suppose $\{\alpha_d^{(0)}\}$ is non-identical and $x^{(0)}>0$. From (2.18) there exist some $k\in\mathscr{D}$ such that $\alpha_k^{(0)}$ is not equal to the aggregated $\alpha^{(0)}$, which in turn implies $\alpha_k^{(1)} \neq \alpha_k^{(0)}$ according to (2.19).

The *aggregated* equilibrium refers to the situation that the aggregated inclination and adoption rate do not change from where the process starts. Mathematically, (α, x) is an aggregated equilibrium of (2.22) if

$$(\alpha^{(t)}, x^{(t)}) = (\alpha, x) \quad \text{for all} \quad t \geq 0.$$

An individual equilibrium $(\{\alpha_d\}, x)$ always has a corresponding aggregated equilibrium (α, x), while the converse is not true. It might be the case that the aggregated inclination is constant but individual inclinations change over time. However, this change can only happen once (if it happens) from period 0 to period 1. In other words, from $t = 0$ to $t = 1$ the initial inclinations $\{\alpha_d^{(0)}\}$ change to $\{\alpha_d^{(1)}\} \neq \{\alpha_d^{(0)}\}$ while $\alpha^{(1)} = \alpha^{(0)}$. For $t \geq 2$ one has $\alpha_d^{(t)} = \alpha_d^{(t-1)}$ for all $d \in \mathscr{D}$, because from (2.19) it follows that if aggregated inclination $\alpha^{(t)}$ and adoption rate $x^{(t)}$ stay constant over time so does $\alpha_d^{(t)}$. Note that the discussion here is about properties of aggregated equilibrium where existence is assumed. More necessary conditions on existence will be discussed in the subsequent section, while deriving sufficient conditions is too complicated for general settings.

The above arguments can be summarised as follows:

1. An individual equilibrium always induces an aggregated equilibrium.
2. Any non-trivial individual equilibrium is identical.
3. Non-trivial aggregated equilibria are of two kinds. The corresponding individual inclinations
   (a) are identical and thus also constitutes an individual equilibrium; or
   (b) change only once from period 0 to period 1 and stay unchanged thereafter.

Since individual equilibria are either trivial or identical, both of which are not interesting, we focus on the aggregated process $(\alpha^{(t)}, x^{(t)})$ hereafter.

Remark 2.3. If we let $\tau_d = 0$ for $d \in \mathscr{D}$ in (2.19), individual inclinations become memoryless which are simply the link-average inclination of the previous period, i.e.,

$$\alpha_d^{(t)} = \alpha^{(t-1)} = \sum_{d \in \mathscr{D}} \widetilde{P}(d) \alpha_d^{(t-1)} .$$

This leads to an immediate coincidence of individual inclinations to $\alpha^{(0)}$ in the very first period after the process started. On the other hand, if we let $\tau_d \to \infty$, every individual will keep her initial inclination forever and not update at all. Interestingly, the aggregated inclination will be $\alpha^{(t)} = \alpha^{(0)}$ for $t > 0$, which coincides with the case of $\tau_d = 0$. In both cases, the dynamical process is reduced to the one-dimensional system discussed in Section 2.2.1.

The complexity of the dynamical system makes it impossible to find closed form solutions under general settings. In the upcoming sections we discuss properties of (2.22) for some special cases of networks and costs. The following section discusses networks with a small number of possible degrees.

2.3.3 Existence of non-trivial equilibrium

Trivial equilibria always exist and are not interesting because no one adopts B. The existence of non-trivial identical equilibria, however, depends on whether there are non-zero intersections of the 45° line and the function $1 - \sum_{d\in\mathscr{D}} \widetilde{P}(d)F(1/(g(d)x))$. If $\widehat{x}$ is such a non-zero intersection, then any identical initial inclination profile $\{\alpha_d^{(0)}\}$ where $\alpha_d^{(0)} = \widehat{\alpha}$ for all $d \in \mathscr{D}$ together with $\widehat{x}$ is an equilibrium.

Here we derive some necessary conditions for existence of non-trivial and non-identical (thus aggregated) equilibria of system (2.22), and provide examples of such equilibria, for networks with a small number of possible degrees. The following Proposition 2.2 shows a necessary condition of existence of non-trivial and non-identical aggregated equilibria when there are only two different degrees in the network.

Proposition 2.2. *Suppose the set of possible degrees is given by* $\mathscr{D} = \{d_1, d_2\}$, *and* $0 < \tau_{d_1} < \infty$, $0 < \tau_{d_2} < \infty$. *Non-trivial and non-identical aggregated equilibria exist only if* $\tau_{d_1} \neq \tau_{d_2}$ *and* $\tau_{d_1}/\tau_{d_2} = d_1/d_2$.

Proof. See Section 2.5 Appendix. □

The next proposition shows a necessary condition of existence of non-trivial and non-identical equilibria in three-degree networks.

Proposition 2.3. *Suppose the set of possible degrees is given by* $\mathscr{D} = \{d_1, d_2, d_3\}$. *Without loss of generality, we let* $d_1 < d_2 < d_3$. *Non-trivial and non-identical aggregated equilibria exist only if* $\tau_d = \tau > 0$ *for all* $d \in \mathscr{D}$ *and* $\alpha_d^{(0)}$, $d \in \mathscr{D}$, *is not weakly monotonic in* d.[11]

Proof. See Section 2.5 Appendix. □

It gets too complicated to derive the sufficient conditions of existence of equilibrium. Nevertheless, it is possible to check whether there exists an equilibrium through the following steps.

[11] Here weakly monotonic means either $\alpha_{d_1}^{(0)} \le \alpha_{d_2}^{(0)} \le \alpha_{d_3}^{(0)}$ or $\alpha_{d_1}^{(0)} \ge \alpha_{d_2}^{(0)} \ge \alpha_{d_3}^{(0)}$.

1. Let $\mathscr{D} = \{d_1, \ldots, d_D\}$, and set parameters $\{\widetilde{P}(d)\}$, F, $\{\tau_d\}$, and β.
2. Set values for $\alpha_{d_3}^{(0)}, \ldots, \alpha_{d_D}^{(0)}$ as given, and take $\alpha_{d_1}^{(0)}$, $\alpha_{d_2}^{(0)}$ and $x^{(0)}$ as variables.
3. Aggregated inclinations $\alpha^{(0)}$ and $\alpha^{(1)}$ can be calculated by (2.18) and (2.19) as functions of $\alpha_{d_1}^{(0)}$, $\alpha_{d_2}^{(0)}$ and $x^{(0)}$. In order to have an aggregated equilibrium, i.e. $\alpha^{(0)} = \alpha^{(1)}$, it should hold

$$\sum_{d \in \mathscr{D}} \widetilde{P}(d) \alpha_d^{(0)} = \sum_{d \in \mathscr{D}} \widetilde{P}(d) \frac{\tau_d \alpha_d^{(0)} + d x^{(0)} \sum_{k \in \mathscr{D}} \widetilde{P}(k) \alpha_k^{(0)}}{\tau_d + d x^{(0)}}. \tag{2.23}$$

4. Since $x^{(t)}$ depends on the values of $x^{(t-1)}$ and $\{\alpha_d^{(t-1)}\}$, and $\{\alpha_d^{(0)}\} \neq \{\alpha_d^{(1)}\}$, to make $x^{(0)}$ an equilibrium state it needs to satisfy $x^{(1)} = x^{(0)}$ and $x^{(2)} = x^{(0)}$, which are

$$x^{(1)} = 1 - \sum_{d \in \mathscr{D}} \widetilde{P}(d) F\Big(\frac{1}{d^\beta \alpha_d^{(0)} x^{(0)}}\Big) = x^{(0)}, \tag{2.24}$$

$$x^{(2)} = 1 - \sum_{d \in \mathscr{D}} \widetilde{P}(d) F\Big(\frac{1}{d^\beta \alpha_d^{(1)} x^{(1)}}\Big) = x^{(0)}, \tag{2.25}$$

respectively.

5. By noting that all the three equations (2.23), (2.24) and (2.25) contain only functions of variables $\alpha_{d_1}^{(0)}$, $\alpha_{d_2}^{(0)}$ and $x^{(0)}$, the solution of these three equations, if exists, together with $\alpha_{d_3}^{(0)}, \ldots, \alpha_{d_D}^{(0)}$ form an equilibrium.

The above procedure needs to solve a non-linear equation system with three variables, which can be done numerically. If a solution has non-identical inclinations and non-zero adoption rate, it is a non-trivial and non-identical aggregated equilibrium.

Remark 2.4. The equations $x^{(2)} = x^{(1)} = x^{(0)}$ and $\alpha^{(1)} = \alpha^{(0)}$ guarantee that $x^{(t)} = x^{(0)}$ and $\alpha^{(t)} = \alpha^{(0)}$ for all $t > 1$. That is why we only have three independent equations and thus only three variables. It is also possible to set $x^{(0)}$ as given and choose three individual inclinations as variables when $D \geq 3$.

Example 2.6. Consider a network with degrees $\mathscr{D} = \{1, 2, 3\}$ represented by $\widetilde{P}(1) = \frac{1}{4}$, $\widetilde{P}(2) = \frac{1}{2}$, $\widetilde{P}(3) = \frac{1}{4}$. Let F be uniform on $[0, 10]$, and set $\beta = 1$, $\tau_1 = 1$, $\tau_2 = 2$, and $\tau_3 = 3$. We take $\alpha_3^{(0)} = 10$, and numerically solve (2.23), (2.24) and (2.25) with the Matlab `fsolve` function with initial values $\alpha_1^{(0)} = 11, \alpha_2^{(0)} = 12$ and $x^{(0)} = 0.5$. The solution is

$$(\alpha_1^{(0)}, \alpha_2^{(0)}, x^{(0)}) = (12.354,\ 13.104,\ 0.069),$$

resulting in an aggregated equilibrium $(\alpha^*, x^*) = (12.140,\ 0.069)$.

2.3.4 Long run behavior and convergence: non-monotonicity

The present section is intended to illustrate the dynamics of (2.22). Naturally, it is possible that the trajectory of $(\{\alpha_d^{(t)}\}, x^{(t)})$ converges to a limit $(\{\alpha_d^*\}, x^*)$ as t increases. We refer to the corresponding aggregated level (α^*, x^*) as an *attractor* [12]. To be precise, a pair $(\alpha^*, x^*) \in \mathbb{R}_+ \times [0,1]$ is an attractor of (2.22) if there exists a trajectory $(\{\alpha_d^{(t)}\}, x^{(t)})$ such that

$$\lim_{t\to\infty} (\alpha^{(t)}, x^{(t)}) = (\alpha^*, x^*),$$

where $\alpha^{(t)} = \sum_{d\in\mathcal{D}} \alpha_d^{(t)}$. Because of the complexity of the system (2.22), it is impossible to obtain any useful conclusion from theoretical analysis. In what follows we look into numerical simulations on the long run behavior of aggregated $(\alpha^{(t)}, x^{(t)})$.

We focus on two aspects: existence of non-trivial attractors, and non-monotonic behavior of aggregated adoption rate $x^{(t)}$. To keep the model as simple as possible, we take $\beta = 0$ and assume the cost distribution F be uniform on $[0, b]$. In this special setting (2.20) becomes

$$x^{(t)} = 1 - \sum_{d\in\mathcal{D}} \min\{1, \frac{1}{b\alpha_d^{(t-1)} x^{(t-1)}}\} \widetilde{P}(d) = \sum_{d\in\mathcal{D}} \widetilde{P}(d) \Big[1 - \min\{1, \frac{1}{b\alpha_d^{(t-1)} x^{(t-1)}}\}\Big]$$

We choose a scale free network $P(d) \propto d^{-2.5}$, where the symbol $\propto$ means "proportional to", with $\mathcal{D} = \{1, \dots, 100\}$. The learning flexibility parameters τ_d are taken as identical, i.e. $\tau_d = \tau$ for all $d \in \mathcal{D}$. The dynamics are qualitatively similar under different values of τ and b, so we only show the results of $\tau = 0.2$ and $b = 10$. The initial levels of individual inclinations have a subtle impact on the dynamics. We firstly consider a series $\{\alpha_d^{(0)}\}$ in which $\alpha_d^{(0)}$ is decreasing in d, and call it the default setting. Later on we will compare the result with that of increasing $\alpha_d^{(0)}$.

Figure 2.6 depicts some points of $(\alpha^{(t)}, x^{(t)})$, represented by dots, and their moving directions, represented by arrows, on the (x, α)-plane for $t = 0, 1, 2, 3, 5$ and 10 under the default setting. The initial points are taken as a two dimensional 21×21 grid. The (m, n)-th point has initial adoption rate $x^{(0)} = 0.05(m-1)$ so that all initial adoption rates are equally distributed over $[0, 1]$, and initial individual inclination $\alpha_d^{(0)} = \sum_{r\in\mathcal{D}} \widetilde{P}(r) \sum_{k=1}^{r} P(k) - \sum_{k=1}^{d} P(k) + 0.1(n-1) + 0.2$ so that the corresponding aggregated inclination $\alpha^{(0)} = 0.1(n-1) + 0.2$ which is equally distributed over $[0.2, 2.2]$. Note that every initial value of individual inclination is greater than zero, and thus it is greater than zero as well at time $t > 0$. It is easy to see that $\alpha_d^{(0)}$ is decreasing in d. Here it is worthwhile to mention that these plots are not normal vector

[12] In general dynamical systems, an attractor is usually also an equilibrium state. But in our model (2.22), a state that the system converges to is not necessarily an equilibrium. The name attractor only describes the convergence, i.e., it has corresponding basin of attraction.

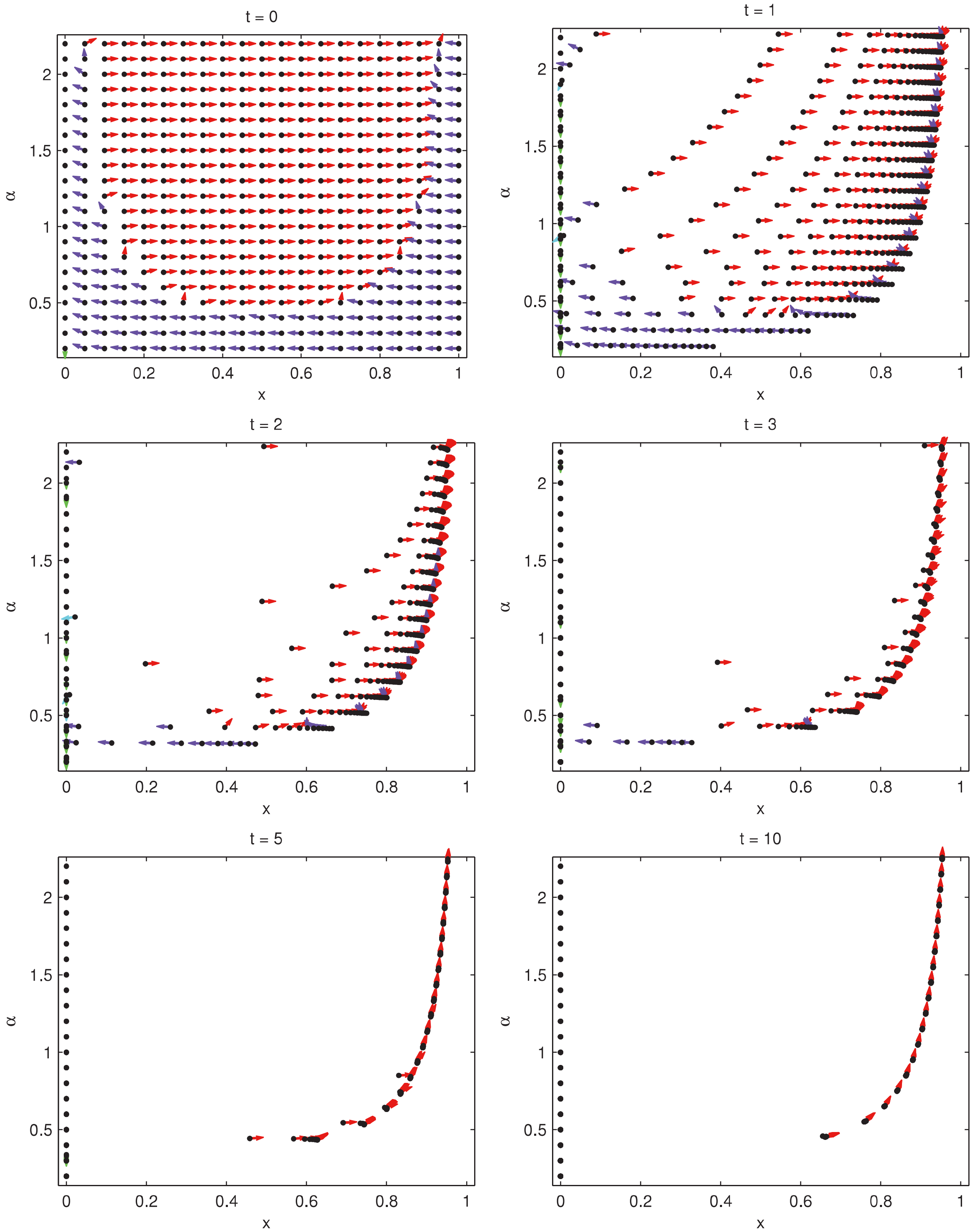


Figure 2.6: Some points and their moving directions of $(\alpha^{(t)}, x^{(t)})$ on the (x,α)-plane for $t = 0, 1, 2, 3, 5, 10$ under the default setting. Different directions are emphasized by different colors. In these figures, ↗ indicates increase in both x and α, ↓ indicates no change in x and decrease in α, ↙, indicates decrease in both x and α, and ↖ indicates decrease in x and increase in α.

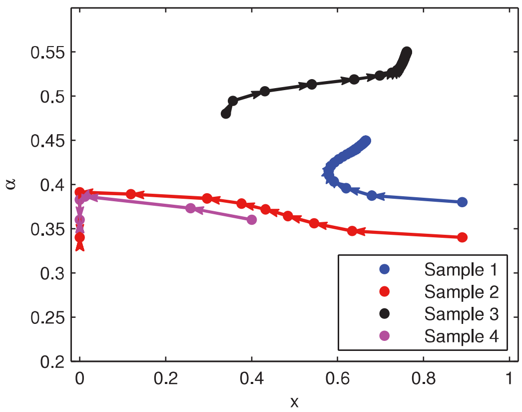
field plots of dynamical systems. Moreover, system (2.22) cannot be described by a normal vector field plot since the initial inclinations $\alpha_d^{(0)}$ are embedded into the system with non-vanishing weights. It means that a same position may have different moving directions at different time t, and a position with a time $t > 0$ may not have corresponding initial values. The last remark on these plots is that the arrows do not represent distances that the current points are going to move. This setting is mainly due to the readability of the graphs.

One observes that at $t = 0$, all aggregated inclinations are increasing except the points with zero adoption rate. For all the graphs, the points with non-zero adoption rate can be divided into groups by the moving directions of adoption rate $x^{(t)}$, represented by $\nearrow$ arrows and $\nwarrow$ arrows. It is as if there is a U-shaped function $f(x)$ which divides the (x, α)-plane such that the points above $f(x)$ get their adoption rates increased, and the points below $f(x)$ get their adoption rates decreased. Unfortunately, we are unable to prove the existence and find the closed form expression of $f(x)$. It can be seen that some points eventually reach zero adoption level and then stay at state $(\alpha^{(0)}, 0)$. The others seem to move along the unverified function $f(x)$ after some periods. This includes some ones which are initially located below $f(x)$.

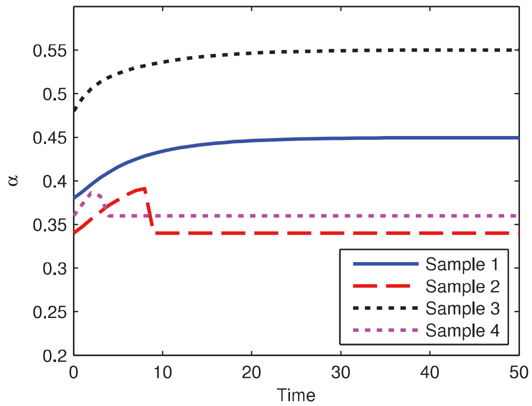
The existence of non-trivial attractors, i.e., the attractors with non-zero adoption rate, can be verified in Figure 2.7. Four sample paths with different initial states are illustrated in 2.7-(a) through 2.7-(c). Of particular interest is sample 1 which exhibits non-monotonic evolution of adoption rate and converges to a non-trivial attractor. The aggregated inclination is increasing all the time in this sample path. This behavior is similar to what has been observed in Figure 2.5, where parameter $\alpha^{(t)}$ was considered to be exogenously given and set to be increasing in time. Although increasing inclination levels are necessary for reaching a non-trivial attractor, it is not sufficient. This can be seen from sample 2 which starts at a slightly lower aggregated inclination comparing to sample 1, but eventually converges to a trivial attractor. In this case the increase of inclination is not quick enough to catch the dropping adoption rate back on an growing track. Figure 2.7-(d) depicts the basins of attraction [13] of trivial and non-trivial attractors.

To demonstrate that the non-monotonic behavior of adoption rate is not a special case, we now consider another setting which has different initial individual inclinations. We again consider a 21×21 grid on the (x, α)-plane, where the (m, n)-th point are now associated with initial individual inclinations $\alpha_d^{(0)} = 0.01d + 0.02n + 0.2$ so that $\{\alpha_d^{(0)}\}$ is increasing in d. All the other parameters are unchanged. We call this the contrast setting. Figure 2.8 depicts the points and their moving directions for $t = 0, 1, 2, 3, 5$ and 10. The aggregated inclinations are decreasing almost all the time, and there are areas of increasing adoption rate and decreasing

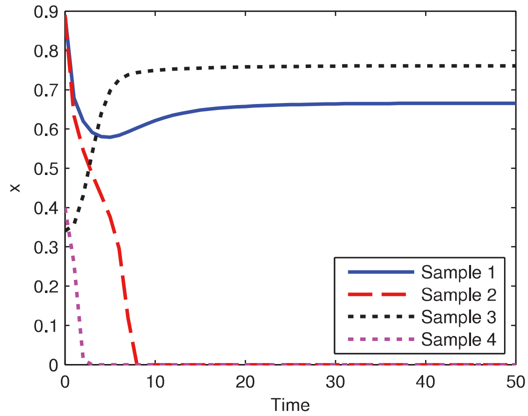
[13] The set of initial states which lead the long run behavior of a dynamical system to an attractor.



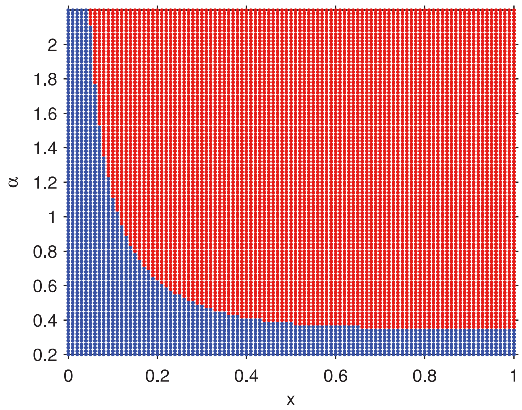
(a) Sample paths starting with different initial values



(b) Dynamical behavior of aggregated inclination



(c) Dynamical behavior of aggregated adoption rate



(d) Basins of attraction: blue/red points indicate the initial states that converge to attractors with zero/non-zero adoption rate.

Figure 2.7: Sample paths and basins of attraction under the default setting

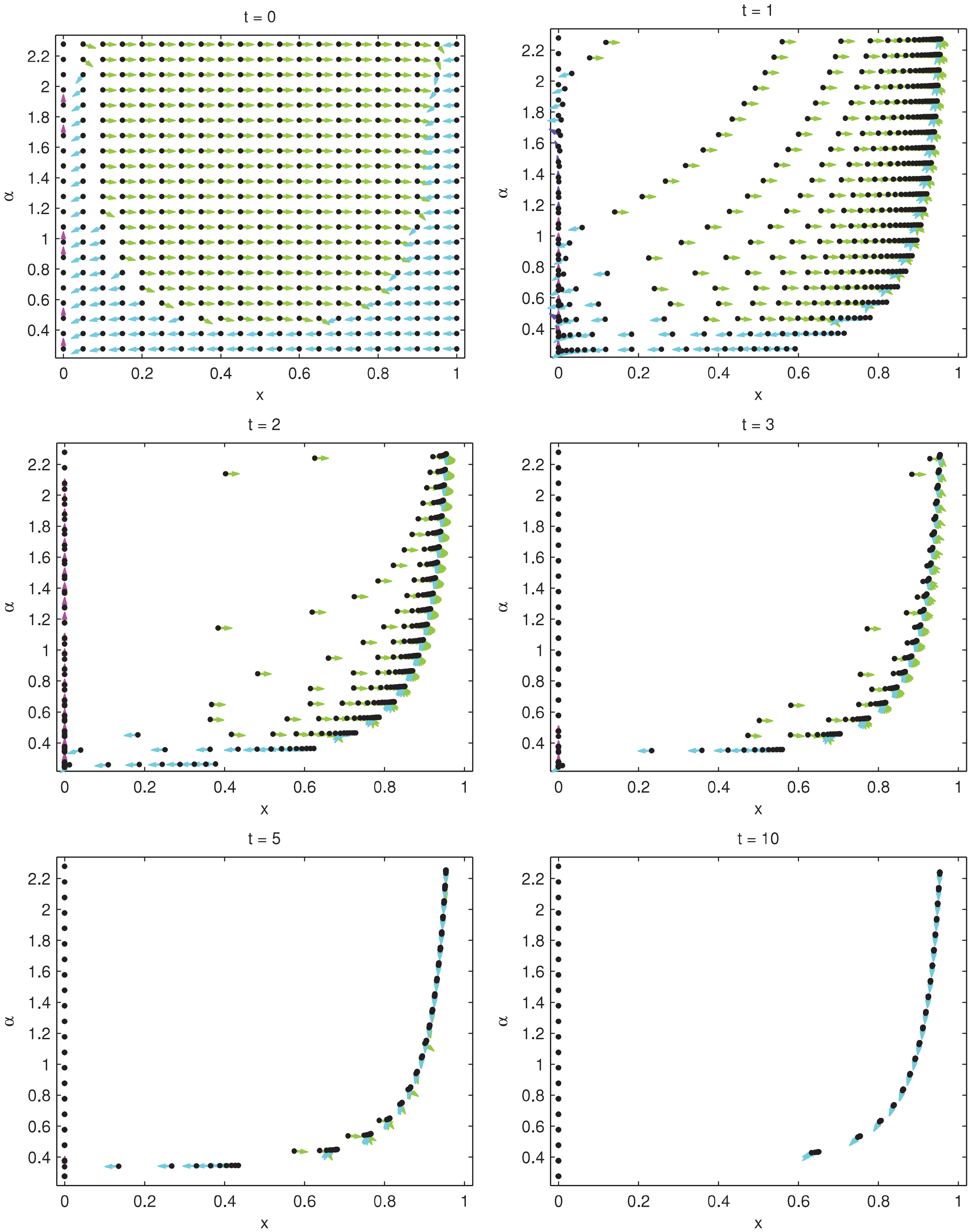
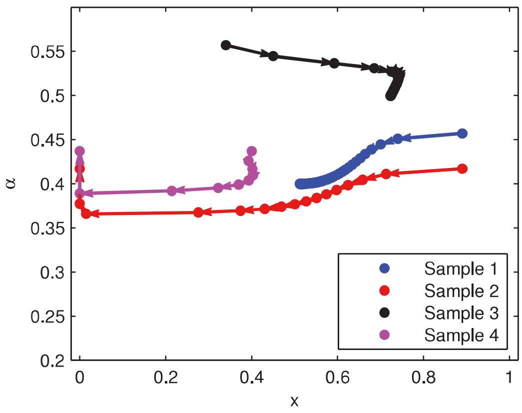
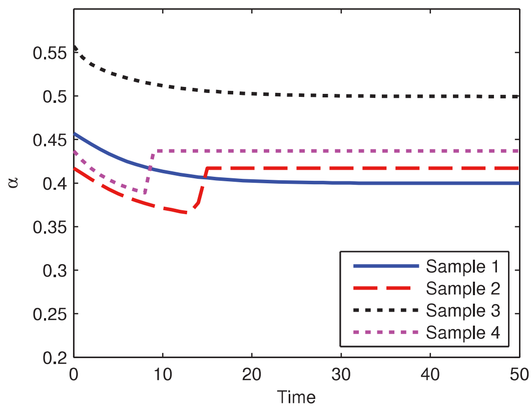


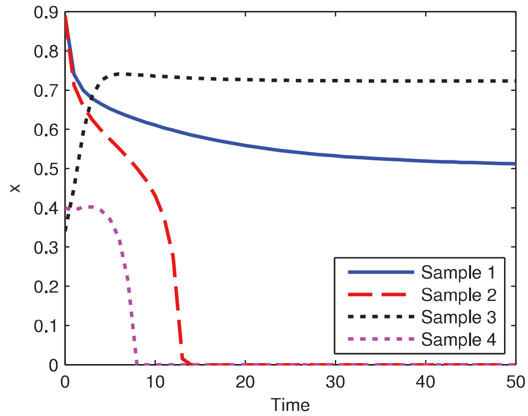
Figure 2.8: Some points and their moving directions of $(\alpha^{(t)}, x^{(t)})$ on the (x,α)-plane for $t = 0, 1, 2, 3, 5, 10$ under the contrast setting. Different directions are emphasized by different colors. In these figures, $\searrow$ indicates increase in x and decrease in α, $\swarrow$, indicates decrease in both x and α, and $\nwarrow$ indicates decrease in x and increase in α, and $\uparrow$ indicates no change in x and increase in α.



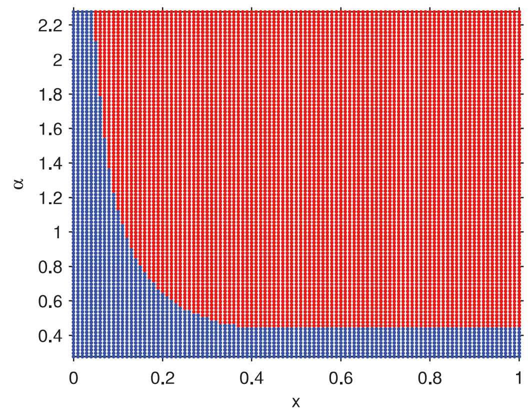
(a) Sample paths starting with different initial values



(b) Dynamical behavior of aggregated inclination



(c) Dynamical behavior of aggregated adoption rate



(d) Basins of attraction: blue/red points indicate the initial states that converge to attractors with zero/non-zero adoption rate.

Figure 2.9: Sample paths and basins of attraction under the contrast setting

adoption rate which seem to be separated by an unverified function. Non-trivial attractors are also observed.

Sample paths plots that are parallel to Figure 2.7 is shown in Figure 2.9. Here we can observe the non-monotonic evolution of aggregated adoption rate from sample 3, where the adoption rate is increasing at the beginning and turns to decrease afterward. The aggregated inclination is decreasing all the time.

We summarize the long run behavior of the aggregated inclination and adoption rate as follows:

- Non-trivial attractors are observed as well as trivial ones.
- The behavior of aggregated adoption rate can be non-monotonic both under the de-

fault and the contrast settings.

- Different initial states lead to different attractors.

- There seems to be an unverified function $f(x)$ on the (x,α)-plane which distinguishes the moving direction of adoption rate on period 0.

2.4 Concluding remarks

In this chapter we proposed a network game model where the driving factor describing the inclination towards adopting a new behavior is endogenous. The learning dynamics about the inclinations is governed by the underlying social network. The most important observation is that the aggregated adoption rate can display non-monotonic changes in its evolution. This could help to understand "sparks" of collective action, i.e., sudden change in behavior as if the collectivity was conducted. The explanation is that collective behavior is coupled to social learning, the invisible part, which governs by means of the social network. The traditional definition of tipping points is not appropriate as its level depends on inclinations and therefore changes as time goes on. Despite the complexity of the model, we are able to analyze necessary conditions of equilibria for networks with low dimensionality. For networks with a larger number of possible degrees we have seen by simulation that aggregated adoption rate and inclination show qualitatively different trajectories for different starting points. The fact that the network was given by a degree distribution instead of an adjacency matrix made extensive simulation possible without running into storage problems.

2.5 Appendix: proofs of propositions

2.5.1 Proof of Proposition 2.2

We first assume $\tau_{d_1} = \tau_{d_2} = \tau$ and relax it later. Suppose (α, x) is an aggregated equilibrium of (2.22) with $x > 0$. Since $d \in \{d_1, d_2\}$, aggregated inclination α satisfies the following simplified equation

$$\alpha = \widetilde{P}(d_1)\frac{\tau\alpha_{d_1}^{(0)} + d_1 x\alpha}{\tau + d_1 x} + \widetilde{P}(d_2)\frac{\tau\alpha_{d_2}^{(0)} + d_2 x\alpha}{\tau + d_2 x}\ . \tag{2.26}$$

It follows that

$$\begin{aligned}
\alpha &= \frac{\widetilde{P}(d_1)(\tau\alpha^{(0)}_{d_1} + d_1 x\alpha)(\tau + d_2 x)}{(\tau + d_1 x)(\tau + d_2 x)} + \frac{\widetilde{P}(d_2)(\tau\alpha^{(0)}_{d_2} + d_2 x\alpha)(\tau + d_1 x)}{(\tau + d_1 x)(\tau + d_2 x)} \\
&= \frac{\widetilde{P}(d_1)\big[\tau^2\alpha^{(0)}_{d_1} + \tau x(d_1\alpha + d_2\alpha^{(0)}_{d_1}) + x^2\alpha d_1 d_2\big]}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\
&\quad + \frac{\widetilde{P}(d_2)\big[\tau^2\alpha^{(0)}_{d_2} + \tau x(d_2\alpha + d_1\alpha^{(0)}_{d_2}) + x^2\alpha d_1 d_2\big]}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\
&= \frac{\tau^2\alpha^{(0)} + \tau x\big[\widetilde{P}(d_1)(d_1\alpha + d_2\alpha^{(0)}_{d_1}) + \widetilde{P}(d_2)(d_2\alpha + d_1\alpha^{(0)}_{d_2})\big] + x^2\alpha d_1 d_2}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\
&= \frac{\tau^2\alpha^{(0)} + \tau x\big[\{\widetilde{P}(d_1)d_1 + \widetilde{P}(d_2)d_2\}\alpha + \{\widetilde{P}(d_1)d_2\alpha^{(0)}_{d_1} + \widetilde{P}(d_2)d_1\alpha^{(0)}_{d_2}\}\big] + x^2\alpha d_1 d_2}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\
&= \alpha + E_1
\end{aligned}$$

where

$$\begin{aligned}
E_1 &= \frac{\tau^2(\alpha^{(0)} - \alpha)}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\
&\quad + \frac{\tau x\big[\{(\widetilde{P}(d_1) - 1)d_1 + (\widetilde{P}(d_2) - 1)d_2\}\alpha + \{\widetilde{P}(d_1)d_2\alpha^{(0)}_{d_1} + \widetilde{P}(d_2)d_1\alpha^{(0)}_{d_2}\}\big]}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\
&= \frac{\tau^2(\alpha^{(0)} - \alpha) + \tau x\big[\{-\widetilde{P}(d_2)d_1 - \widetilde{P}(d_1)d_2\}\alpha + \{\widetilde{P}(d_1)d_2\alpha^{(0)}_{d_1} + \widetilde{P}(d_2)d_1\alpha^{(0)}_{d_2}\}\big]}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\
&= \frac{\tau^2(\alpha^{(0)} - \alpha) + \tau x\big[\widetilde{P}(d_1)d_2(\alpha^{(0)}_{d_1} - \alpha) + \widetilde{P}(d_2)d_1(\alpha^{(0)}_{d_2} - \alpha)\big]}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2}\,.
\end{aligned}$$

Therefore it holds that $E_1 = 0$. Since $\tau > 0$, $d_1 > 0, d_2 > 0$, and $x \neq 0$, one has

$$\begin{aligned}
& \tau^2(\alpha^{(0)} - \alpha) + \tau x\big[\widetilde{P}(d_1)d_2(\alpha^{(0)}_{d_1} - \alpha) + \widetilde{P}(d_2)d_1(\alpha^{(0)}_{d_2} - \alpha)\big] = 0 \\
\Leftrightarrow\quad & \tau^2\alpha^{(0)} + \tau x\big[\widetilde{P}(d_1)d_2\alpha^{(0)}_{d_1} + \widetilde{P}(d_2)d_1\alpha^{(0)}_{d_2}\big] = \tau^2\alpha + \tau x\big[\widetilde{P}(d_1)d_2 + \widetilde{P}(d_2)d_1\big]\alpha \\
\Leftrightarrow\quad & \alpha = \frac{\tau^2\alpha^{(0)} + \tau x\big[\widetilde{P}(d_1)d_2\alpha^{(0)}_{d_1} + \widetilde{P}(d_2)d_1\alpha^{(0)}_{d_2}\big]}{\tau^2 + \tau x\big[\widetilde{P}(d_1)d_2 + \widetilde{P}(d_2)d_1\big]} \\
& \quad = \alpha^{(0)} + \frac{\tau x\big[\widetilde{P}(d_1)d_2(\alpha^{(0)}_{d_1} - \alpha^{(0)}) + \widetilde{P}(d_2)d_1(\alpha^{(0)}_{d_2} - \alpha^{(0)})\big]}{\tau^2 + \tau x\big[\widetilde{P}(d_1)d_2 + \widetilde{P}(d_2)d_1\big]}\,.
\end{aligned}$$

Here we use a small trick. $\alpha_{d_1}^{(0)} - \alpha^{(0)}$ can be rewritten as

$$\begin{aligned}\alpha_{d_1}^{(0)} - \alpha^{(0)} &= \alpha_{d_1}^{(0)} - \widetilde{P}(d_1)\alpha_{d_1}^{(0)} - \widetilde{P}(d_2)\alpha_{d_2}^{(0)} = \{1-\widetilde{P}(d_1)\}\alpha_{d_1}^{(0)} - \widetilde{P}(d_2)\alpha_{d_2}^{(0)} \\ &= \widetilde{P}(d_2)(\alpha_{d_1}^{(0)} - \alpha_{d_2}^{(0)})\ .\end{aligned}$$

Similarly,

$$\alpha_{d_2}^{(0)} - \alpha^{(0)} = \widetilde{P}(d_1)(\alpha_{d_2}^{(0)} - \alpha_{d_1}^{(0)})\ .$$

Thus we have

$$\begin{aligned}\alpha &= \alpha^{(0)} + \frac{\tau x\big[\widetilde{P}(d_1)d_2\widetilde{P}(d_2)(\alpha_{d_1}^{(0)} - \alpha_{d_2}^{(0)}) + \widetilde{P}(d_2)d_1\widetilde{P}(d_1)(\alpha_{d_2}^{(0)} - \alpha_{d_1}^{(0)})\big]}{\tau^2 + \tau x\big[\widetilde{P}(d_1)d_2 + \widetilde{P}(d_2)d_1\big]} \\ &= \alpha^{(0)} + \frac{\tau x(d_2-d_1)\widetilde{P}(d_1)\widetilde{P}(d_2)(\alpha_{d_1}^{(0)} - \alpha_{d_2}^{(0)})}{\tau^2 + \tau x\big[\widetilde{P}(d_1)d_2 + \widetilde{P}(d_2)d_1\big]}\end{aligned}$$

Since α is an equilibrium state, one has $\alpha^{(0)} = \alpha$, which means the second term on the right-hand side of the above equation is 0. This can only happen when $\alpha_{d_1}^{(0)} = \alpha_{d_2}^{(0)}$, implying that under $\tau_{d_1} = \tau_{d_2}$ non-identical and non-trivial aggregated equilibrium does not exist.

For the case of $\tau_{d_1} \neq \tau_{d_2}$, the same procedure can be used. After some algebra we obtain the following equation

$$\alpha = \alpha^{(0)} + \frac{x(d_2\tau_{d_1} - d_1\tau_{d_2})\widetilde{P}(d_1)\widetilde{P}(d_2)(\alpha_{d_1}^{(0)} - \alpha_{d_2}^{(0)})}{\tau_{d_1}\tau_{d_2} + x\big[\widetilde{P}(d_1)d_2\tau_{d_1} + \widetilde{P}(d_2)d_1\tau_{d_2}\big]}.$$

Since $\alpha = \alpha^{(0)}$, the second term on the right-hand side of the above equation becomes zero, which implies either $\alpha_{d_1}^{(0)} = \alpha_{d_2}^{(0)}$ or $d_2\tau_{d_1} = d_1\tau_{d_2}$. Therefore $\tau_{d_1}/\tau_{d_2} = d_1/d_2$ is a necessary condition for existence of non-identical and non-trivial aggregated equilibrium under $\tau_{d_1} \neq \tau_{d_2}$. □

2.5.2 Proof of Proposition 2.3

The similar strategy as that in the proof of Proposition 2.2 is used, thus we only show an outline of analysis but ignore the details of algebra. Suppose (α, x) is an aggregated equilibrium of (2.22) with $x > 0$. Then α satisfies the following equation.

$$
\begin{aligned}
\alpha &= \sum_{d\in\{d_1,d_2,d_3\}} \widetilde{P}(d)\frac{\tau\alpha_d^{(0)} + dx\alpha}{\tau + d\alpha} \\
&= \widetilde{P}(d_1)\frac{\tau\alpha_{d_1}^{(0)} + d_1 x\alpha}{\tau + d_1 x} + \widetilde{P}(d_2)\frac{\tau\alpha_{d_2}^{(0)} + d_2 x\alpha}{\tau + d_2 x} + \widetilde{P}(d_3)\frac{\tau\alpha_{d_3}^{(0)} + d_3 x\alpha}{\tau + d_3 x}
\end{aligned} \tag{2.27}
$$

By adding the fractions into one fraction and simplifying it, one has

$$\alpha = \alpha + \frac{E_2^N}{E_2^D},$$

where

$$
\begin{aligned}
E_2^N = {} & \tau^3(\alpha^{(0)} - \alpha) \\
& + \tau^2 x\Big[d_1\big\{\widetilde{P}(d_2)(\alpha_{d_2}^{(0)} - \alpha) + \widetilde{P}(d_3)(\alpha_{d_3}^{(0)} - \alpha)\big\} \\
& \qquad + d_2\big\{\widetilde{P}(d_3)(\alpha_{d_3}^{(0)} - \alpha) + \widetilde{P}(d_1)(\alpha_{d_1}^{(0)} - \alpha)\big\} \\
& \qquad + d_3\big\{\widetilde{P}(d_1)(\alpha_{d_1}^{(0)} - \alpha) + \widetilde{P}(d_2)(\alpha_{d_2}^{(0)} - \alpha)\big\}\Big] \\
& + \tau x^2\Big[d_1 d_2\widetilde{P}(d_3)(\alpha_{d_3}^{(0)} - \alpha) + d_2 d_3\widetilde{P}(d_1)(\alpha_{d_1}^{(0)} - \alpha) + d_3 d_1\widetilde{P}(d_2)(\alpha_{d_2}^{(0)} - \alpha)\Big]
\end{aligned}
$$

and

$$E_2^D = (\tau + d_1 x)(\tau + d_2 x)(\tau + d_3 x) > 0\,.$$

Then we obtain $E_2^N = 0$. Solving it for α yields $\alpha = \alpha^{(0)} + \dfrac{E_3^N}{E_3^D}$ where

$$
\begin{aligned}
E_3^N = \tau x\Big[& (\tau + d_3 x)(d_2 - d_1)\widetilde{P}(d_1)\widetilde{P}(d_2)(\alpha_{d_1}^{(0)} - \alpha_{d_2}^{(0)}) \\
& + (\tau + d_1 x)(d_3 - d_2)\widetilde{P}(d_2)\widetilde{P}(d_3)(\alpha_{d_2}^{(0)} - \alpha_{d_3}^{(0)}) \\
& + (\tau + d_2 x)(d_3 - d_1)\widetilde{P}(d_1)\widetilde{P}(d_3)(\alpha_{d_1}^{(0)} - \alpha_{d_3}^{(0)})\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
E_3^D = {} & \tau^3 + \tau^2 x\Big[d_1\big\{\widetilde{P}(d_2) + \widetilde{P}(d_3)\big\} + d_2\big\{\widetilde{P}(d_3) + \widetilde{P}(d_1)\big\} + d_3\big\{\widetilde{P}(d_1) + \widetilde{P}(d_2)\big\}\Big] \\
& + \tau x^2\Big[d_1 d_2\widetilde{P}(d_3) + d_2 d_3\widetilde{P}(d_1) + d_3 d_1\widetilde{P}(d_2)\Big] > 0\,.
\end{aligned}
$$

Since α is the aggregated inclination in equilibrium, it holds $\alpha = \alpha^{(0)}$ which then implies

$E_3^N = 0$. Note that E_3^N contains a sum of three terms which cannot be all positive or negative. So we need a careful discussion for different initial values of $\{\alpha_{d_1}^{(0)}, \alpha_{d_2}^{(0)}, \alpha_{d_3}^{(0)}\}$. Non-identical initial inclinations can be categorized as the following.

$$
\begin{array}{ll}
1) & \alpha_{d_1}^{(0)} = \alpha_{d_2}^{(0)} < \alpha_{d_3}^{(0)} \Rightarrow E_3^N < 0 \\
& \alpha_{d_1}^{(0)} = \alpha_{d_2}^{(0)} > \alpha_{d_3}^{(0)} \Rightarrow E_3^N > 0 \\
& \alpha_{d_1}^{(0)} = \alpha_{d_3}^{(0)} \neq \alpha_{d_2}^{(0)} \Rightarrow E_3^N = 0 \text{ if } \dfrac{\tau + d_3 x}{\tau + d_1 x} = \dfrac{(d_3 - d_2)\widetilde{P}(d_3)}{(d_2 - d_1)\widetilde{P}(d_1)} \\
& \alpha_{d_2}^{(0)} = \alpha_{d_3}^{(0)} < \alpha_{d_1}^{(0)} \Rightarrow E_3^N > 0 \\
& \alpha_{d_2}^{(0)} = \alpha_{d_3}^{(0)} > \alpha_{d_1}^{(0)} \Rightarrow E_3^N < 0
\end{array}
$$

$$
\begin{array}{ll}
2) & \alpha_{d_1}^{(0)} < \alpha_{d_2}^{(0)} < \alpha_{d_3}^{(0)} \Rightarrow E_3^N < 0 \\
& \alpha_{d_1}^{(0)} < \alpha_{d_3}^{(0)} < \alpha_{d_2}^{(0)} \rightarrow \text{see the discussion below} \\
& \alpha_{d_2}^{(0)} < \alpha_{d_1}^{(0)} < \alpha_{d_3}^{(0)} \rightarrow \text{see the discussion below} \\
& \alpha_{d_2}^{(0)} < \alpha_{d_3}^{(0)} < \alpha_{d_1}^{(0)} \rightarrow \text{see the discussion below} \\
& \alpha_{d_3}^{(0)} < \alpha_{d_1}^{(0)} < \alpha_{d_2}^{(0)} \rightarrow \text{see the discussion below} \\
& \alpha_{d_3}^{(0)} < \alpha_{d_2}^{(0)} < \alpha_{d_1}^{(0)} \Rightarrow E_3^N > 0
\end{array}
$$

The proof of Proposition 2.3 completes here. However, we would like to discuss more about necessary conditions when $\alpha_d^{(0)}$'s are all different and non-monotonic.

Here we only focus on the case $\alpha_{d_2}^{(0)} < \alpha_{d_1}^{(0)} < \alpha_{d_3}^{(0)}$, since the discussion for other cases is similar. From $E_3^N = 0$ one has

$$
\begin{aligned}
&(\tau + d_3 x)(d_2 - d_1)\widetilde{P}(d_1)\widetilde{P}(d_2)(\alpha_{d_1}^{(0)} - \alpha_{d_2}^{(0)}) \\
&= (\tau + d_1 x)(d_3 - d_2)\widetilde{P}(d_2)\widetilde{P}(d_3)(\alpha_{d_3}^{(0)} - \alpha_{d_2}^{(0)}) + (\tau + d_2 x)(d_3 - d_1)\widetilde{P}(d_1)\widetilde{P}(d_3)(\alpha_{d_3}^{(0)} - \alpha_{d_1}^{(0)}).
\end{aligned}
$$

Both sides of the above equation are larger then 0. Solving this equation for $\alpha_{d_3}^{(0)}$ leads to $\alpha_{d_3}^{(0)} = C_1 \alpha_{d_1}^{(0)} + C_2 \alpha_{d_2}^{(0)}$ where

$$
C_1 = \frac{(\tau + d_3 x)(d_2 - d_1)\widetilde{P}(d_1)\widetilde{P}(d_2) + (\tau + d_2 x)(d_3 - d_1)\widetilde{P}(d_1)\widetilde{P}(d_3)}{(\tau + d_1 x)(d_3 - d_2)\widetilde{P}(d_2)\widetilde{P}(d_3) + (\tau + d_2 x)(d_3 - d_1)\widetilde{P}(d_1)\widetilde{P}(d_3)}, \text{ and}
$$

$$
C_2 = \frac{(\tau + d_1 x)(d_3 - d_2)\widetilde{P}(d_2)\widetilde{P}(d_3) - (\tau + d_3 x)(d_2 - d_1)\widetilde{P}(d_1)\widetilde{P}(d_2)}{(\tau + d_1 x)(d_3 - d_2)\widetilde{P}(d_2)\widetilde{P}(d_3) + (\tau + d_2 x)(d_3 - d_1)\widetilde{P}(d_1)\widetilde{P}(d_3)}.
$$

It is easy to see that $C_1 + C_2 = 1$, $C_1 > 0$ and $C_2 < 1$. This means $\alpha_{d_3}^{(0)}$ is an affine combination of $\alpha_{d_1}^{(0)}$ and $\alpha_{d_2}^{(0)}$. Since $\alpha_{d_2}^{(0)} < \alpha_{d_1}^{(0)} < \alpha_{d_3}^{(0)}$, one has $C_1 > 1$. Equivalently,

$$
\begin{aligned}
& (\tau + d_3 x)(d_2 - d_1)\widetilde{P}(d_1)\widetilde{P}(d_2) > (\tau + d_1 x)(d_3 - d_2)\widetilde{P}(d_2)\widetilde{P}(d_3) \\
\Leftrightarrow \quad & \frac{\tau + d_3 x}{\tau + d_1 x} > \frac{(d_3 - d_2)\widetilde{P}(d_3)}{(d_2 - d_1)\widetilde{P}(d_1)} \,. \qquad (2.28)
\end{aligned}
$$

(2.28) is a necessary condition for existence of equilibrium when $\alpha_{d_2}^{(0)} < \alpha_{d_1}^{(0)} < \alpha_{d_3}^{(0)}$. □